Preparation for the Final.

Below you can find a list of definitions, axioms (well, an axiom), theorems and (counter-)examples that you need to know for the Final. More precisely, for in-class part of the Final, you will be given several (tentatively, ten to twenty) items from this list to formulate. Note that it won't have to be word-by-word citation, but whatever you write will need to be (a) correct (as in not a false statement), (b) easily equivalent to the textbook/lectures version.

On in-class part of the Final, you will *not* be asked to provide any proofs.

Knowledge of proofs of statements on this list is required for the take-home part of the Final.

In the list below,
$\odot$ marks definitions;
$\square$ marks theorems and statements;
$\triangleright$ marks (counter-)examples that you need to know off-hand.

Note that there may be corrections to this list until Friday, Dec 11.

Preliminaries.

- $\odot$ Sets, set-theoretic operations, functions, inverse function, composition.
- $\odot$ Injections, surjections, bijections.
- $\odot$ Finite, infinite sets. Denumerable (countably infinite), countable, uncountable sets.
- $\square$ Countability of $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^n$, $\mathbb{Q}$.
- $\square$ Cantor's Theorem. Uncountability of $\mathbb{R}$.

Properties of $\mathbb{R}$.

- $\odot$ Bounded, bounded above, bounded below subsets of $\mathbb{R}$.
- $\odot$ Upper bound, lower bound of a subset of $\mathbb{R}$.
- $\odot$ Least upper bound (= exact upper bound = supremum) of a subset of $\mathbb{R}$.
- $\odot$ Greatest lower bound (= exact lower bound = infimum) of a subset of $\mathbb{R}$.
- $\odot$ Completeness property of $\mathbb{R}$ (= supremum property of $\mathbb{R}$).
- $\square$ Archimedean property of $\mathbb{R}$.
- $\square$ Nested intervals property.
- $\square$ The density theorem.

Limits of Sequences.

- $\odot$ Sequence of real numbers (= sequence in $\mathbb{R}$).
- $\odot$ Limit of a sequence in $\mathbb{R}$, convergent/divergent sequence.
- $\square$ Uniqueness of limit of a sequence.
- $\odot$ Tail of a sequence.
- $\odot$ Bounded sequence.

- ☐ Boundedness of a convergent sequence.
- ▷ Bounded but divergent sequence.
- ☐ Arithmetic properties of limits of sequences (Theorem 3.2.3).
- ▷ Divergent sequences A, B such that $A + B$ converges.
- ☐ Order properties of limits of sequences (Theorems 3.2.4, 3.2.5).
- ▷ Sequences $(a_n), (b_n)$ with $a_n > b_n$ for all $n \in \mathbb{N}$, but $\lim(a_n) = \lim(b_n)$.
- ☐ Squeeze theorem for sequences.
- ☐ Increasing, strictly increasing, decreasing, strictly decreasing, monotone sequence.
- ☐ Monotone convergence theorem.
- ⊙ Euler's number e.
- ☐ Square root as a limit of a sequence.
- ⊙ Subsequence of a sequence.
- ☐ Bolzano–Weierstrass theorem.
- ⊙ Cauchy sequence.
- ☐ Cauchy criterion.
- ⊙ Sequence that tends to $+\infty$, sequence that tends to $-\infty$, properly divergent sequence.
- ⊙ Sum of a series. Correspondence between infinite series and sequences.
- ☐ n-th term test.

Limits of Functions.

- ⊙ Cluster point of a subset of $\mathbb{R}$.
- ⊙ Limit of a function.
- ☐ Uniqueness of limit of a function.
- ☐ Sequential criterion for limit of a function.
- ⊙ Function bounded a neighborhood.
- ☐ Local boundedness of a function that has a limit.
- ▷ Bounded function that does not have a limit at 0.
- ☐ Arithmetic properties of limits of functions (Theorem 4.2.4).
- ▷ Functions f, g that don't have a limit at some point $c \in \mathbb{R}$, but $f + g$ does.
- ☐ Order properties of limits of functions (Theorem 4.2.6).
- ▷ Functions f, g such that for all x in their domain, $f > g$, but at some point c, $\lim\limits_{x \to c} f = \lim\limits_{x \to c} g$.
- ☐ Squeeze theorem for limits of functions.
- ☐ Local separation from zero (Theorem 4.2.9).
- ⊙ Infinite limit of a function, limit of a function at infinity, infinite limit of a function at infinity.

- ⊙ One-sided limits.

Continuous Functions.

- ⊙ Function, continuous at a point. Function, discontinuous at a point.
- □ Criterion for continuity in terms of neighborhoods (Theorem 5.1.2).
- □ Sequential criterion for continuity.
- □ Sequential criterion for discontinuity.
- ⊙ Function, continuous on a subset of $\mathbb{R}$.
- ▷ Function, discontinuous everywhere (for example, Dirichlet's function).
- ▷ Function, continuous at irrational numbers and discontinuous at rational numbers (for example, Thomae's function).
- □ Arithmetic properties of continuous functions (Theorem 5.2.1).
- ▷ Functions f, g discontinuous at 0 such that $f + g$ is continuous at 0.
- □ Composition of continuous functions (at a point and on a set).
- □ Boundedness Theorem.
- ▷ Bounded but discontinuous (at least at one point) function.
- ▷ Function continuous but unbounded on an open interval.
- ⊙ Absolute (= global) maximum of a function on a set, point of absolute maximum. Absolute minimum of a function on a set, point of absolute minimum.
- □ Maximum–Minimum Theorem.
- ▷ Function f continuous and bounded on an open interval such that that f has neither maximum nor minimum value.
- □ Location of roots theorem, Bolzano's intermediate value theorem.
- □ Preservation of closed intervals. Preservation of intervals.
- ⊙ Increasing, strictly increasing, decreasing, strictly decreasing, monotone functions.
- ⊙ Jump of a monotone function.
- □ Continuity criterion of monotone functions (Theorem 5.6.3).
- □ Continuous inverse theorem. Continuity of nth root function.
- □ Definition and basic properties of the rational power function $x^r, r \in \mathbb{Q}$.
- ⊙ Function, uniformly continuous on a subset of $\mathbb{R}$.
- □ Uniform continuity theorem.

Differentiation.

- ⊙ Derivative of a function at a point. Function, differentiable at a point. (Three definitions: limit of ratio, Caratheodory's, linear approximation.)
- □ Continuity of a differentiable function.
- ▷ Function, continuous but not differentiable at $x = 0$.
- ▷ Function, differentiable on $\mathbb{R}$, whose derivative is not continuous at 0.

- ▷ Function, differentiable on $[-1,1]$, whose derivative is not bounded on $[-1,1]$.
- □ Arithmetic properties of derivative.
- □ Chain rule.
- □ Derivative of inverse function (inverse function theorem).
- □ Interior extremum theorem.
- □ Rolle's theorem.
- □ Mean value theorem.
- □ First derivative test for extrema (Theorem 6.2.8).
- □ Criterion for a differentiable function to be increasing/decreasing/constant on an interval (Theorems 6.2.5, 6.2.7).
- ⊙ nth Taylor polynomial of a function.
- □ Taylor's theorem.
- □ nth derivative test for extrema (Theorem 6.4.4).
- □ nth Taylor's polynomial at zero for $(1+x)^\alpha$ ($\alpha \in \mathbb{R}$), e^x, $\ln(1+x)$, $\sin x$, $\cos x$.
- □ Newton's Method for approximating a zero of a function. Recursively defined sequence converging to $\sqrt{2}$.

The Riemann Integral.

- ⊙ Partition, tagged partition, Riemann sum.
- ⊙ Riemann integrable function, Riemann integral.
- ▷ Bounded but not a Riemann integrable function.
- □ Arithmetic and order properties of Riemann integral (linearity and monotonicity).
- □ Boundedness theorem for Riemann integrable function.
- □ Riemann integrability of a continuous function, monotone function.
- □ Interval additivity theorem (proof not required).
- □ The fundamental theorem of calculus (first form).
- ⊙ Indefinite integral, antiderivative.
- □ The fundamental theorem of calculus (second form).
- □ Derivative of an indefinite integral of a continuous function.
- □ Substitution theorem.
- □ Definition and basic properties of the logarithmic, exponential functions.
- □ Definition and basic properties of the power function.
- □ Definition and basic properties of sine and cosine. (If covered on 12/10.)